

Uniformly Rotating Homogeneous Rings in post-Newtonian Gravity

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ABSTRACT

In this paper uniformly rotating relativistic rings are investigated analytically utilizing two different approximations simultaneously: (1) an expansion about the thin ring limit (the cross-section is small compared with the size of the whole ring) (2) post-Newtonian expansions. The analytic results for rings are compared with numerical solutions.

Key words: gravitation – methods: analytical – stars: rotation.

1 INTRODUCTION

The problem of self-gravitating, axially symmetric rings in equilibrium can be tackled in Newtonian gravity by expanding it about the thin ring limit, where the cross-section of the ring tends to a circle. Doing so, Kowalewsky (1885), Poincaré (1885a,b,c) and Dyson (1892, 1893) obtained series for homogeneous rings and Ostriker (1964a,b, 1965) and Petroff & Horatschek (2008) for polytropic rings. Alternatively by using a Roche model (Roche 1873) rings with a sufficiently soft equation of state can be described approximately, see Horatschek & Petroff (2009).

With the help of numerical methods it was possible to study homogeneous rings and their connection to the Maclaurin spheroids (Wong 1974; Eriguchi & Sugimoto 1981; Eriguchi & Hachisu 1985; Ansorg, Kleinwächter & Meinel 2003c) as well as non-homogeneous rings (Hachisu 1986). Furthermore relativistic rings and their transition to the extreme Kerr Black Hole (Ansorg, Kleinwächter & Meinel 2003b; Fischer, Horatschek & Ansorg 2005; Labranche, Petroff & Ansorg 2007) can be calculated to near-machine accuracy.

However no analytic work has been done for relativistic rings. The reason may be that because of the non-linearity of Einstein’s field equations the expansion about the thin ring limit does not work. In contrast, if one additionally expands the rings in a post-Newtonian series, following the methods used for studying post-Newtonian Maclaurin spheroids (Chandrasekhar 1967; Bardeen 1971; Petroff 2003), it does. In this paper the first post-Newtonian corrections for homogeneous rings are calculated and the results are compared to numerical ones generated using a version of the code

described in Ansorg et al. (2003a), but modified to rings, cf. Meinel et al. (2008).

2 BASIC EQUATIONS

The matter model is that of a perfect fluid, i.e. the energy-momentum tensor reads

$$T^{ik} = pg^{ik} + \left(\mu + \frac{p}{c^2}\right)u^i u^k, \quad (1)$$

where p is the pressure, c the speed of light, μc^2 the energy density and u^i the four-velocity. We are considering homogeneous matter, by which we mean the mass-density is constant:

$$\mu = \text{constant}. \quad (2)$$

An axially symmetric, stationary spacetime containing (only) a rigidly rotating fluid can be described using the line element

$$ds^2 = e^{2\alpha}(d\varrho^2 + dz^2) + \varrho^2 e^{2\kappa}(d\varphi - \omega dt)^2 - e^{2\nu} c^2 dt^2, \quad (3)$$

where the metric functions depend only on ϱ and z . The coordinates are uniquely fixed by requiring that the metric functions and their first derivatives be continuous at the fluid surface. We introduce the further coordinates r and χ (see Fig. 1)

$$\varrho = b - r \cos \chi, \quad z = r \sin \chi, \quad (4)$$

where b is the ‘centre of mass’ of the ring’s cross-section, defined by

$$b := \frac{\iint \mu \varrho d\varrho dz}{\iint \mu d\varrho dz} \quad (5)$$

and we have retained μ in this and the next equation in this paper to have expressions valid for the more general case in

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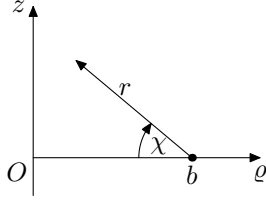


Figure 1. A sketch providing the meaning of the coordinates (r, χ) .

which μ is not constant. The parameter b is a coordinate dependent quantity and has no particular physical significance in contrast to invariants such as mass or angular momentum. Inserting (4) in the numerator of (5), one finds that

$$\iint \mu r^2 \cos \chi \, dr \, d\chi = 0 \quad (6)$$

holds. Since we are considering rigid rotation, we can introduce the constant angular velocity $\Omega = u^\varphi / u^t$. A locally non-rotating observer would measure the following three-velocity for a fluid element

$$v := \varrho(\Omega - \omega)e^{\kappa-\nu}, \quad (7)$$

see Bardeen (1970).

It is convenient to introduce the following dimensionless quantities

$$\tilde{v} := \frac{v}{c}, \quad \tilde{\omega} := \frac{\omega}{\Omega}, \quad \tilde{\Omega} := \frac{b\Omega}{c}, \quad \tilde{p} := \frac{p}{\mu c^2}. \quad (8)$$

The field equations are

$$\begin{aligned} \Delta\nu &= \frac{4\pi G\mu e^{2\alpha}}{c^2} \left[\frac{1+\tilde{v}^2}{1-\tilde{v}^2}(1+\tilde{p}) + 2\tilde{p} \right] \\ &\quad - \nabla\nu \cdot \nabla(\kappa + \nu) + \frac{\varrho^2 \tilde{\Omega}^2}{2b^2} e^{2(\kappa-\nu)} (\nabla\tilde{\omega})^2, \end{aligned} \quad (9)$$

$$\Delta_1(\kappa + \nu) = \frac{16\pi G\mu}{c^2} \tilde{p} e^{2\alpha} - [\nabla(\kappa + \nu)]^2, \quad (10)$$

$$\begin{aligned} \Delta_2\tilde{\omega} &= -\frac{16\pi G\mu}{c^2} \frac{(1-\tilde{\omega})(1+\tilde{p})}{1-\tilde{v}^2} e^{2\alpha} \\ &\quad - \nabla\tilde{\omega} \cdot \nabla(3\kappa - \nu), \end{aligned} \quad (11)$$

$$\begin{aligned} \Delta_{-1}\alpha &= -\frac{4\pi G\mu}{c^2} (1+\tilde{p}) e^{2\alpha} + \nabla\kappa \cdot \nabla\nu \\ &\quad + \frac{\varrho^2}{4b^2} e^{2(\kappa-\nu)} (\nabla\tilde{\omega})^2 + \frac{1}{\varrho} \frac{\partial\nu}{\partial\varrho}, \end{aligned} \quad (12)$$

where G is the gravitational constant. Here ∇ is the nabla operator in a three-space with the cylindrical coordinates (ϱ, φ, z) and Δ_m is a generalized Laplacian operator, which when applied to an axially symmetric function $f = f(\varrho, z)$ gives

$$\Delta_m f = \frac{\partial^2 f}{\partial\varrho^2} + \frac{m+1}{\varrho} \frac{\partial f}{\partial\varrho} + \frac{\partial^2 f}{\partial z^2}. \quad (13)$$

The operator is linear and becomes the standard Laplacian for $m = 0$, i.e. $\Delta_0 = \Delta$. The vanishing divergence of the energy-momentum tensor, $T^{ik}{}_{;k} = 0$, gives

$$(1+\tilde{p})e^\nu \sqrt{1-\tilde{v}^2} = 1 - \gamma, \quad (14)$$

where γ is a constant related to the relative redshift Z_0 via

$$Z_0 = \frac{\gamma}{1-\gamma}. \quad (15)$$

To be more precise, this is the redshift of photons emitted from the ring's surface and observed at infinity that carry no angular momentum.

3 POST-NEWTONIAN EXPANSION

Let us introduce the relativistic parameter

$$\varepsilon := \frac{\sqrt{G\mu}b}{c} \quad (16)$$

and the following expansions

$$\nu = \sum_{i=1}^{\infty} \nu_{2i} \varepsilon^{2i}, \quad \kappa = \sum_{i=1}^{\infty} \kappa_{2i} \varepsilon^{2i}, \quad (17)$$

$$\alpha = \sum_{i=1}^{\infty} \alpha_{2i} \varepsilon^{2i}, \quad \tilde{\omega} = \sum_{i=1}^{\infty} \tilde{\omega}_{2i} \varepsilon^{2i}, \quad (18)$$

$$\gamma = \sum_{i=1}^{\infty} \gamma_{2i} \varepsilon^{2i}, \quad \tilde{\Omega} = \sum_{i=1}^{\infty} \tilde{\Omega}_{2i-1} \varepsilon^{2i-1}, \quad (19)$$

$$\tilde{v} = \sum_{i=1}^{\infty} \tilde{v}_{2i-1} \varepsilon^{2i-1}, \quad \tilde{p} = \sum_{i=1}^{\infty} \tilde{p}_{2i} \varepsilon^{2i}, \quad (20)$$

where the even/odd powers of ε that appear in the sums come about from stationarity (simultaneous reversal in the direction of time $dt \rightarrow -dt$ and in the direction of rotation $\Omega \rightarrow -\Omega$). The surface of the fluid, defined by vanishing pressure, will be parameterized by the function $r = r_s(\chi)$, which we also expand

$$r_s(\chi) = \sum_{i=0}^{\infty} r_{2i}(\chi) \varepsilon^{2i}. \quad (21)$$

When one replaces $G\mu/c^2$ in the field equations by ε^2/b^2 and expands them in ε , then the leading order equation comes from (9) and is the familiar Laplace/Poisson equation (in a somewhat unfamiliar notation)

$$\Delta\nu_2^o = 0, \quad \Delta\nu_2^i = \frac{4\pi}{b^2}, \quad (22)$$

and the expansion of (14) yields the Bernoulli equation

$$\tilde{p}_2 - \frac{\tilde{v}_1^2}{2} + \nu_2^i = -\gamma_2, \quad (23)$$

where $\tilde{v}_1 = \varrho\tilde{\Omega}_1/b$ results from (7) and the superscripts ‘i’ and ‘o’ refer to the regions inside and outside the fluid respectively. To leading order, (6) becomes

$$\int_0^{2\pi} r_0^3(\chi) \cos \chi \, d\chi = 0. \quad (24)$$

On the Newtonian boundary $r_0(\chi)$, $\tilde{p}_2 = 0$ holds and the function ν_2 and its first derivatives must be continuous. It follows from asymptotic flatness that ν_2 tends to zero at infinity. No analytic solution for rings is known even in the Newtonian case (described by the above equations). Instead, we solve them using an expansion about the thin ring limit as

in Petroff & Horatschek (2008). For example, the expansion for the Newtonian surface function is

$$r_0(\chi) = a \left(1 + \sum_{i=1}^q \sum_{k=1}^i \beta_{ik} \cos(k\chi) \sigma^i + o(\sigma^q) \right) \quad (25)$$

with

$$\sigma := \frac{a}{b} \quad (26)$$

and where it will be convenient to introduce

$$\lambda := \ln \frac{8}{\sigma} - 2 \quad (27)$$

for later. The expression for r_0 makes use of the fact that the cross-section tends to a circle of radius a for $\sigma \rightarrow 0$, as mentioned in the introduction. The solution given for Newtonian rings in the above paper can be used here, where the relations

$$\begin{aligned} U &\equiv c^2 \varepsilon^2 \nu_2, & \Omega_N &\equiv cb^{-1} \varepsilon \tilde{\Omega}_1, & V_0 &\equiv -c^2 \varepsilon^2 \gamma_2, \\ p_N &\equiv \mu c^2 \varepsilon^2 \tilde{p}_2, & v_N &\equiv c \varepsilon \tilde{v}_1 \end{aligned} \quad (28)$$

have to be taken into account and the index ‘N’ denotes a Newtonian quantity here to distinguish it from the relativistic quantity.

Turning now to the first post-Newtonian correction, we find the equations

$$\Delta_1(\kappa_2 + \nu_2) = 0, \quad (29)$$

$$\Delta_{-1}(\alpha_2 + \nu_2) = 0, \quad (30)$$

$$\Delta_2 \tilde{\omega}_2^o = 0, \quad (31)$$

$$\Delta_2 \tilde{\omega}_2^i = -\frac{16\pi}{b^2}, \quad (32)$$

$$\Delta \nu_4^o = 0, \quad (33)$$

$$\Delta \nu_4^i = \frac{8\pi}{b^2} \left(-\nu_2^i + \tilde{v}_1^2 + \frac{3}{2} \tilde{p}_2 \right). \quad (34)$$

The first two of these equations are solved by

$$\kappa_2 = \alpha_2 = -\nu_2, \quad (35)$$

where we remind the reader that ν_2 , \tilde{p}_2 and \tilde{v}_1 are already known from the Newtonian order (cf. (28)) and in (34) α_2^i was replaced by $-\nu_2^i$. Equation (14) gives

$$\begin{aligned} \tilde{p}_4 + \tilde{p}_2 \left(\nu_2^i - \frac{\tilde{v}_1^2}{2} \right) + \nu_4^i \\ + \frac{\nu_2^{i2}}{2} - \frac{1}{2} \nu_2^i \tilde{v}_1^2 - \frac{\tilde{v}_1^4}{8} - \tilde{v}_1 \tilde{v}_3 = -\gamma_4, \end{aligned} \quad (36)$$

where

$$\tilde{v}_3 = \frac{\varrho}{b} \left[\tilde{\Omega}_3 - \tilde{\Omega}_1(\tilde{\omega}_2^i + 2\nu_2^i) \right] \quad (37)$$

follows from the expansion of (7). Equation (6) yields

$$\int_0^{2\pi} r_2(\chi) r_0(\chi)^2 \cos \chi \, d\chi = 0. \quad (38)$$

Additionally, the transition conditions at the fluid boundary become

$$\left(\tilde{\omega}_2^i - \tilde{\omega}_2^o \right)_{r=r_0(\chi)} = 0, \quad (39)$$

$$\left. \frac{\partial(\tilde{\omega}_2^i - \tilde{\omega}_2^o)}{\partial r} \right|_{r=r_0(\chi)} = 0, \quad (40)$$

$$\left(\nu_4^i - \nu_4^o \right)_{r=r_0(\chi)} = 0, \quad (41)$$

$$\left. \frac{\partial(\nu_4^i - \nu_4^o)}{\partial r} \right|_{r=r_0(\chi)} + r_2 \left. \frac{\partial^2(\nu_2^i - \nu_2^o)}{\partial r^2} \right|_{r=r_0(\chi)} = 0. \quad (42)$$

at this order. Vanishing pressure on the surface translates to

$$\tilde{p}_4|_{r=r_0(\chi)} + r_2(\chi) \left. \frac{\partial \tilde{p}_2}{\partial r} \right|_{r=r_0(\chi)} = 0. \quad (43)$$

4 EXPANSION ABOUT THE THIN RING LIMIT

As with the Newtonian order, we expand the unknown functions as a power series in σ and a Fourier series in χ (the sine terms do not appear as a result of reflectional symmetry with respect to the equatorial plane):

$$\nu_4^i = \pi^2 \sigma^4 \sum_{i=0}^q \sum_{k=0}^i N_{ik}^i(y) \cos(k\chi) \sigma^i + o(\sigma^{q+4}), \quad (44)$$

$$\nu_4^o = \pi^2 \sigma^4 \sum_{l=1}^{q+1} \sum_{i=l-1}^q F_{li} I_l(y, \chi) a^{2l-1} \sigma^{i-l} + o(\sigma^{q+4}), \quad (45)$$

$$\tilde{\omega}_2^i = \pi \sigma^2 \sum_{i=0}^q \sum_{k=0}^i w_{ik}^i(y) \cos(k\chi) \sigma^i + o(\sigma^{q+2}), \quad (46)$$

$$\tilde{\omega}_2^o = \pi \sum_{l=1}^{q+1} \sum_{i=l-1}^q G_{li} K_l(y, \chi) a^{2l+1} \sigma^{i-l} + o(\sigma^{q+2}) \quad (47)$$

$$\tilde{p}_4 = \pi^2 \sigma^4 \sum_{i=0}^q \sum_{k=0}^i q_{ik}(y) \cos(k\chi) \sigma^i + o(\sigma^{q+4}), \quad (48)$$

$$\tilde{\Omega}_3 = \pi^{3/2} \sigma^2 \sum_{i=1}^{q+1} L_i \sigma^i + o(\sigma^{q+3}), \quad (49)$$

$$\gamma_4 = \pi^2 \sigma^4 \sum_{i=0}^q g_i \sigma^i + o(\sigma^{q+4}), \quad (50)$$

$$\begin{aligned} \frac{r_2}{a} \equiv y_2 = \pi \sigma^2 d + \pi \sigma^2 \sum_{i=1}^q \sum_{k=1}^i \kappa_{ik} \cos(k\chi) \sigma^i \\ + o(\sigma^{q+2}), \end{aligned} \quad (51)$$

where

$$y := \frac{r}{a} \quad (52)$$

and the parameter d introduced in (51) represents the post-Newtonian contribution to the radius of the ring to leading order in σ . The relevance of this parameter will be discussed below. An important property of the Fourier series, is that they terminate due to the expansion in σ . The functions $I_l(y, \chi)$ form an axially symmetric set of solutions to Laplace’s equation, regular everywhere except at $r = 0$ and vanishing at infinity, see Petroff & Horatschek (2008). The set $K_l(y, \chi)$ is the analogue for the generalized Laplace’s equation (31), that can be found using separation of variables in toroidal coordinates (η, ξ, φ) . These are defined by

$$\varrho = \frac{b \sinh \xi}{\cosh \xi - \cos \eta}, \quad (53)$$

$$z = \frac{b \sin \eta}{\cosh \xi - \cos \eta}. \quad (54)$$

The set K_l can be defined by

$$K_1 := \frac{\pi}{\sqrt{2}b^3} \frac{(\cosh \xi - \cos \eta)^{3/2}}{\sinh \xi} P_{-1/2}^1(\cosh \xi), \quad (55)$$

$$K_l := \left(-\frac{1}{b} \frac{d}{db} \right)^{l-1} K_1 \quad (56)$$

with

$$\frac{d}{db} := \frac{\partial}{\partial b} + \cos \chi \frac{\partial}{\partial r} - \frac{\sin \chi}{r} \frac{\partial}{\partial \chi}. \quad (57)$$

On the axis, they have the simple form

$$K_l(r) = -\frac{\pi(2l-1)!!}{4r^{2l+1}} \quad (\text{on axis}). \quad (58)$$

The idea now is to expand these functions with respect to σ and then proceed to solve the field equations iteratively. We demonstrate the technique with the leading order in σ .

5 THE SOLUTION TO THE EQUATIONS

The post-Newtonian correction to the surface to leading order in σ is

$$y_2(\chi) = \pi\sigma^2[d + o(\sigma^0)]. \quad (59)$$

As with y_0 , this function is also independent of χ to leading order and fulfils (38). Inside the ring, (32) becomes

$$\frac{d^2 w_{00}^i}{dy^2} + \frac{1}{y} \frac{dw_{00}^i}{dy} + 16 = 0, \quad (60)$$

whose regular solution is

$$w_{00}^i = -4y^2 + c_1. \quad (61)$$

The transition conditions are satisfied for

$$c_1 = 8\lambda + 4 \quad (62)$$

and where the constant G_{10} from (47) is

$$G_{10} = -16. \quad (63)$$

The generalized Poisson equation (34) becomes

$$\frac{d^2 N_{00}^i}{dy^2} + \frac{1}{y} \frac{dN_{00}^i}{dy} + 20y^2 - 24\lambda - 58 = 0 \quad (64)$$

and we can immediately write down the regular solution

$$N_{00}^i = -\frac{5}{4}y^4 + \left(6\lambda + \frac{29}{2}\right)y^2 + c_2. \quad (65)$$

Here, the transition conditions lead to

$$c_2 = -12\lambda^2 - 54\lambda - 4\lambda d - 8d - \frac{245}{4} \quad (66)$$

and for F_{10} from (45)

$$F_{10} = -12\lambda - 4d - 24. \quad (67)$$

The equilibrium condition (36) gives an expression for the pressure coefficient q_{00} that can be combined with (43) to give

$$q_{00} = \frac{7}{4}y^4 - (4\lambda + 14)y^2 + 4\lambda + \frac{49}{4} + 2d, \quad (68)$$

thus completing the solution to leading order in σ .

In general, stationary and axially symmetric fluids with

a given equation of state are characterized by two parameters, e.g. gravitational mass and angular momentum. The corresponding post-Newtonian expansions contain two additional free parameters, which arise because of the freedom one has in choosing which two quantities contain no post-Newtonian contribution, see Bardeen (1971) and discussion therein. We too would obtain two free parameters, but have already chosen to have b remain unchanged to first order in ε and thus only have d as a free parameter in the solution.¹

The leading order in σ is equivalent to an infinite cylinder, as in the Newtonian case, see Ostriker (1964a,b). In Bičák et al. (2004), relativistic, homogeneous cylinders and their post-Newtonian expansion are investigated. In that paper, there is no post-Newtonian contribution to the central pressure, which corresponds to choosing $d = -2\lambda - 49/8$ in our case. In order to compare our results to theirs, we introduce the proper radial coordinate and circumferential radial coordinate

$$y_p := \frac{1}{a} \int_0^r \sqrt{g_{rr}} dr' = \int_0^y e^\alpha dy', \quad (69)$$

$$y_c := \frac{1}{2\pi a} \int_0^{2\pi} \sqrt{g_{\chi\chi}} d\chi' = ye^\alpha. \quad (70)$$

The values of these coordinates on the cylinder's surface (using the choice for d given above) are found to be

$$\begin{aligned} y_p &= 1 - \frac{35}{24}\pi\sigma^2\varepsilon^2[1 + o(\sigma^0)] + O(\varepsilon^4), \\ y_c &= 1 - \frac{17}{8}\pi\sigma^2\varepsilon^2[1 + o(\sigma^0)] + O(\varepsilon^4), \end{aligned} \quad (\text{on surface}) \quad (71)$$

in agreement with (6.10) and (6.11) in Bičák et al. (2004). The expressions for the pressure, e.g. as a function of proper radial coordinate, can also be shown to agree.

The expansion to higher orders in σ presents no particular difficulties and we have carried it out up to seventh order. The resulting coefficients can be found in Appendix A up to fourth order, but the results presented in the next section are given up to seventh order.

6 RESULTS

In the following it will be convenient to use dimensionless quantities, denoted by a bar:

$$\frac{\bar{\Omega}^2}{\Omega^2} = \frac{1}{G\mu}, \quad \frac{\bar{M}}{M} = \frac{\bar{M}_B}{M_B} = \frac{G^{3/2}\mu^{1/2}}{c^3}, \quad (72)$$

$$\frac{\bar{J}}{J} = \frac{G^2\mu}{c^5}, \quad \frac{\bar{b}}{b} = \frac{\bar{\varrho}}{\varrho} = \frac{\bar{z}}{z} = \frac{G^{1/2}\mu^{1/2}}{c}, \quad (73)$$

where M is the gravitational mass, M_B the baryonic mass and J the angular momentum. In particular

$$\bar{b} = \frac{\sqrt{G\mu}b}{c} = \varepsilon \quad (74)$$

holds. In Table 1, the Newtonian and first post-Newtonian results for the ring with the prescribed values $\varrho_i/\varrho_o = 0.7$

¹ Note that to leading order in σ , one only has the freedom to fix a single parameter. The parameter b does not appear to this order and we retain the freedom to choose d .

Table 1. Comparison between the Newtonian and first post-Newtonian (pN) results up to seventh order in σ . The relative errors are computed by comparing with a highly accurate numerical solution with the prescribed values $\varrho_i/\varrho_o = 0.7$ and $Z_0 = 0.05$. For the definition of d_Ω , d_J and d_M , see (76).

Numerical value	Relative error				
	Relativistic	Newtonian	pN: $d = 0$	pN: $d = d_\Omega$	pN: $d = d_J$ pN: $d = d_M$
$\bar{\Omega} = 4.9108 \times 10^{-1}$		-2×10^{-2}	-1×10^{-3}	-3×10^{-4}	3×10^{-3} 2×10^{-3}
$\bar{J} = 2.3168 \times 10^{-4}$		1×10^{-1}	1×10^{-2}	8×10^{-3}	-8×10^{-3} -2×10^{-3}
$\bar{M} = 7.9661 \times 10^{-3}$		9×10^{-2}	1×10^{-2}	7×10^{-3}	-5×10^{-3} -1×10^{-3}
$\bar{M}_B = 8.0842 \times 10^{-3}$		7×10^{-2}	1×10^{-2}	5×10^{-3}	-4×10^{-3} -1×10^{-3}

and $Z_0 = 0.05$ are compared with a highly accurate numerical solution in full General Relativity. In Newtonian theory, there is (strictly speaking) no redshift, but the leading term, $Z_0 = -V_0/c^2$, contains only the Newtonian quantity V_0 and the speed of light, showing the relativistic origin. This equation explains what is meant by the ‘redshift’ of a Newtonian configuration. The free parameter d is chosen in a number of different ways in the table. For one, we simply choose $d = 0$. Alternatively, we expand a chosen quantity

$$\begin{aligned}\bar{\Omega} &= \bar{\Omega}_N + \bar{\Omega}_{\text{pN}}\varepsilon^2 + O(\varepsilon^4), \\ \bar{J} &= \bar{J}_N + \bar{J}_{\text{pN}}\varepsilon^2 + O(\varepsilon^4), \\ \bar{M} &= \bar{M}_N + \bar{M}_{\text{pN}}\varepsilon^2 + O(\varepsilon^4),\end{aligned}\quad (75)$$

and define d so that there is no post-Newtonian contribution², i.e.

$$\begin{aligned}\bar{\Omega}_{\text{pN}}(d = d_\Omega) &= 0, \\ \bar{J}_{\text{pN}}(d = d_J) &= 0, \\ \bar{M}_{\text{pN}}(d = d_M) &= 0.\end{aligned}\quad (76)$$

The solutions to these equations when expressed as series in σ are

$$\begin{aligned}d_\Omega &= -\frac{144\lambda^2 + 240\lambda + 119}{24(4\lambda + 1)} \\ &\quad + \frac{6912\lambda^3 + 864\lambda^2 - 9684\lambda - 5903}{1152(4\lambda + 1)^2}\sigma^2 + o(\sigma^3) \\ d_J &= -\frac{624\lambda^2 + 3120\lambda + 2009}{24(12\lambda + 7)} \\ &\quad + \frac{202752\lambda^3 + 1487136\lambda^2 + 1765236\lambda + 549479}{1152(12\lambda + 7)^2}\sigma^2 \\ &\quad + o(\sigma^3) \\ d_M &= -3(\lambda + 2) + \frac{15}{64}\sigma^2 + o(\sigma^3).\end{aligned}\quad (77)$$

Throughout Table 1, the post-Newtonian expressions yield a clear improvement compared to the Newtonian ones. The deviations from the accurate numerical values have two origins: Terms in σ and terms in ε are missing. Table 2 demonstrates that for rings with $\varrho_i/\varrho_o \gtrsim 0.7$, $q = 7$ is sufficient for relative errors smaller than 10^{-4} in the Newtonian case. One can estimate that the error due to terminating the series in σ at the order q results in a relative error of about

$(\sigma |\ln \sigma|)^q$, where it is convenient to use

$$\sigma \sim \frac{1}{2} \left(1 - \frac{\varrho_i}{\varrho_o} \right). \quad (78)$$

If the relativistic effects are large in comparison (as is the case in Table 2 although Z_0 is only 0.05), then truncating in σ is negligible. A rough estimate for ε can be found by taking the leading term for the redshift in both ε and σ , inserting (78) and approximating slightly to find

$$\varepsilon \sim \frac{1}{1 - \varrho_i/\varrho_o} \sqrt{\frac{Z_0}{6 - 2\ln(1 - \varrho_i/\varrho_o)}}. \quad (79)$$

The relative factor between the Newtonian and post-Newtonian contribution to Ω is about $5|\ln \sigma|(\sigma\varepsilon)^2 \sim 10^{-2}$ for the values from Table 2.

Tables 3 and 4 present analogous results for $\varrho_i/\varrho_o = 0.95$ and $Z_0 = 10^{-3}$. The values of each expansion parameter (i.e. σ and ε) are small enough that extremely high accuracy is observed. This also permits us to see how the results improve as one proceeds to the post-Newtonian level and with increasing orders in σ . Tables 1 and 3 suggest that the choice $d = d_x$ is particularly accurate if the quantity x is prescribed.

Figure 2 provides an example of the meridional cross-section of the surface shape of a ring. Both post-Newtonian results given there can be seen to constitute a marked improvement as compared to the Newtonian one. Moreover, we find that the choice $d = d_M$ gives a significantly better result than $d = 0$, which is also the case for most values in Tables 1 and 3.

A similar behaviour can be found in Fig. 3, where the pressure in the equatorial plane is plotted. Again the post-Newtonian results demonstrate an improvement as compared to the Newtonian, and here too the choice $d = d_M$ leads to better results than $d = 0$.

The quality of the approximations can be seen for sequences of rings with the prescribed radius ratios $\varrho_i/\varrho_o = 0.95$ (Fig. 4) and $\varrho_i/\varrho_o = 0.7$ (Fig. 5). The relativistic curve ‘begins’ in the Newtonian limit at the origin and ‘ends’ in an extreme Kerr Black Hole at the point (1, 1). The parametric transition of rings to Black Holes was first studied in Ansorg et al. (2003b) and had already been considered analytically for discs (Neugebauer & Meinel 1993; Meinel 2002). It is not surprising that the accuracy of the Newtonian and post-Newtonian curves in the vicinity of this ultra-relativistic point (at which $Z_0 \rightarrow \infty$) is so poor.³ In contrast, the highly

² Nonetheless even for this quantity the values in the Newtonian and post-Newtonian column are different, because $\varrho_i/\varrho_o = 0.7$ and $Z_0 = 0.05$ are prescribed, and not the chosen quantity.

³ It is a mere coincidence that some of the curves almost pass through the point (1, 1).

Table 2. Comparison between the Newtonian and first post-Newtonian (pN) results for the dimensionless angular velocity $\bar{\Omega}$ for different orders q in σ . The prescribed parameters for the ring are $\varrho_i/\varrho_o = 0.7$ and $Z_0 = 0.05$. The numerical calculations show that the value for $\bar{\Omega}$ for the relativistic ring is $\bar{\Omega}_{\text{num}} = 0.49108$ (cf. Table 1) and for the Newtonian one is $\bar{\Omega}_{\text{N,num}} = 0.48109$ (see Table 1 in Ansorg et al. (2003b)). For the definition of d_Ω , d_J and d_M , see (76).

q	Newtonian	pN: $d = 0$	pN: $d = d_\Omega$	pN: $d = d_J$	pN: $d = d_M$
1	0.50085	0.51126	0.51192	0.51364	0.51301
3	0.48274	0.49206	0.49264	0.49423	0.49369
5	0.48135	0.49059	0.49116	0.49277	0.49223
7	0.48114	0.49035	0.49092	0.49255	0.49200
20	0.48109	—	—	—	—

Table 3. Comparison between the Newtonian and first post-Newtonian (pN) results up to seventh order in σ . The relative errors are computed by comparing with a highly accurate numerical solution with the prescribed values $\varrho_i/\varrho_o = 0.95$ and $Z_0 = 10^{-3}$. For the definition of d_Ω , d_J and d_M , see (76).

Numerical value	Relative error				
	Newtonian	pN: $d = 0$	pN: $d = d_\Omega$	pN: $d = d_J$	pN: $d = d_M$
$\bar{\Omega} = 9.6226144 \times 10^{-2}$	-5×10^{-4}	-1×10^{-6}	-3×10^{-7}	8×10^{-7}	7×10^{-7}
$\bar{J} = 2.9191516 \times 10^{-7}$	3×10^{-3}	7×10^{-6}	4×10^{-6}	-2×10^{-6}	-2×10^{-6}
$\bar{M} = 8.5891100 \times 10^{-5}$	1×10^{-3}	5×10^{-6}	2×10^{-6}	-1×10^{-6}	-7×10^{-7}
$\bar{M}_B = 8.5914515 \times 10^{-5}$	1×10^{-3}	4×10^{-6}	2×10^{-6}	-9×10^{-7}	-8×10^{-7}

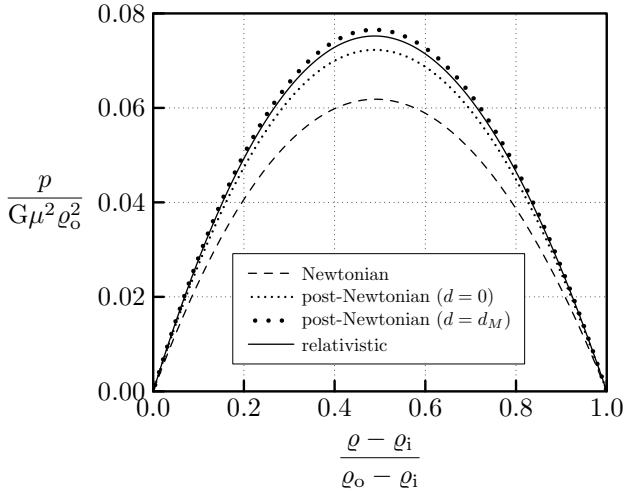


Figure 3. The pressure profile of the ring with $A = 0.7$ and $Z_0 = 0.1$ in the equatorial plane. For the definition of d_Ω , d_J and d_M , see (76).

relativistic regime can be probed quite accurately using very high orders of the post-Newtonian approximation, at least for the disc of dust (Bardeen & Wagoner 1969, 1971; Petroff & Meinel 2001). In the Newtonian regime, $\bar{M}^2/\bar{J} \rightarrow 0$, the curves become indistinguishable, as they must. Naturally, we require that there exist some neighbourhood of the Newtonian limit in which the post-Newtonian approximation provides an improvement compared with the Newtonian results. In order to check this, we have plotted the relative errors $\Delta_{\Omega M} = 1 - (\Omega M)/(\Omega M)_{\text{relativistic}}$ in Figs 6 and 7. The errors corresponding to the sequence $\varrho_i/\varrho_o = 0.95$ shown in Fig. 6 demonstrate the required behaviour. It is particularly apparent that $d = d_M$ produces better results than $d = 0$. For $\varrho_i/\varrho_o = 0.7$, the Newtonian limit shown in Fig. 7 seems to be in contradiction to the requirement: the post-Newtonian approximations in this plot provide no improvement. For

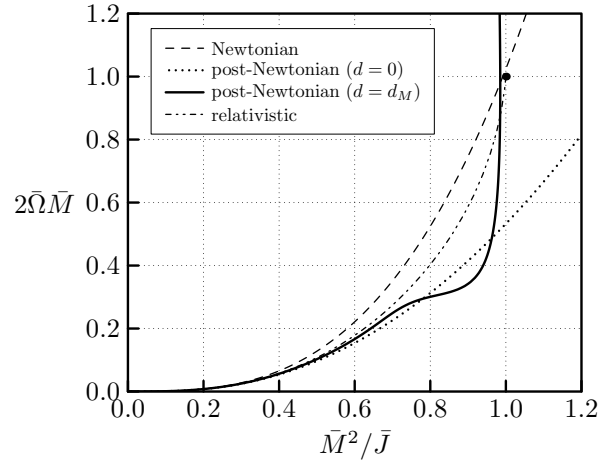


Figure 4. The sequence of rings with $\varrho_i/\varrho_o = 0.95$ for $q = 7$. For the definition of d_Ω , d_J and d_M , see (76).

this relatively small radius ratio, the truncation of the series at the order $q = 7$ provides the limiting value to the accuracy of $\sim 10^{-4}$ as discussed above. On the other hand, the limitation in the relative error of $\sim 10^{-8}$ due to $q = 7$ for $\varrho_i/\varrho_o = 0.95$ is beyond the accuracy shown in Fig. 6.

We have seen, that as with spheroidal bodies, the post-Newtonian approximation for toroidal ones can be used to great advantage. It is noteworthy that this work is possible without the existence of a known exact ring solution, even in Newtonian theory. It was therefore necessary to expand about the thin ring limit, which we were able to do up to high orders, making the results applicable even to fairly ‘thick’ rings.

Table 4. Comparison between the Newtonian and first post-Newtonian (pN) results for the dimensionless angular velocity $\bar{\Omega}$ for different orders q in σ . The prescribed parameters for the ring are $\varrho_i/\varrho_o = 0.95$ and $Z_0 = 10^{-3}$. The numerical calculations show that the value for $\bar{\Omega}$ for the relativistic ring is $\bar{\Omega}_{\text{num}} = 0.096226144$ (cf. Table 3) and for the Newtonian one is $\bar{\Omega}_{\text{N,num}} = 0.096177081$. For the definition of d_Ω , d_J and d_M , see (76).

q	Newtonian	pN: $d = 0$	pN: $d = d_\Omega$	pN: $d = d_J$	pN: $d = d_M$
1	0.096333800	0.096383025	0.096383103	0.096383208	0.096383200
3	0.096177653	0.096226613	0.096226690	0.096226795	0.096226786
5	0.096177085	0.096226044	0.096226121	0.096226226	0.096226217
7	0.096177081	0.096226040	0.096226117	0.096226222	0.096226214
20	0.096177081	—	—	—	—

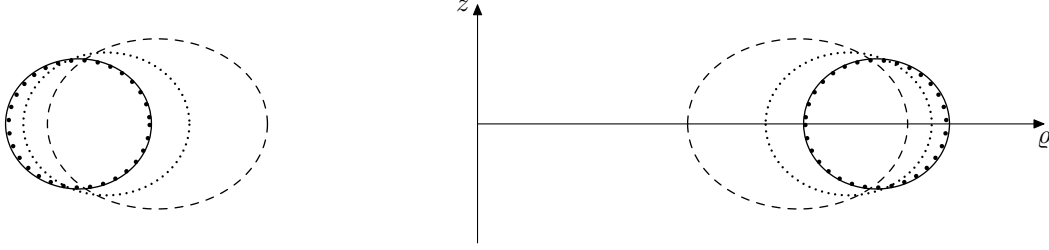


Figure 2. The cross-section of the ring with $\bar{M} = 0.05$ and $Z_0 = 0.2$. The solid line is the highly accurate numerical result. The large dots, which coincide with it almost exactly, represent the post-Newtonian result with $d = d_M$ (cf. (76)). The thin dotted line is the post-Newtonian result with $d = 0$ and the dashed line is the Newtonian result. All analytic curves were generated with $q = 7$.

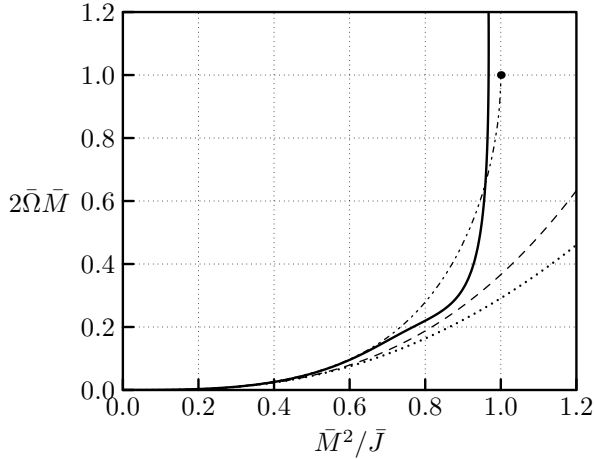


Figure 5. The sequence of rings with $\varrho_i/\varrho_o = 0.7$ for $q = 7$. The line types are the same as in Fig. 4.

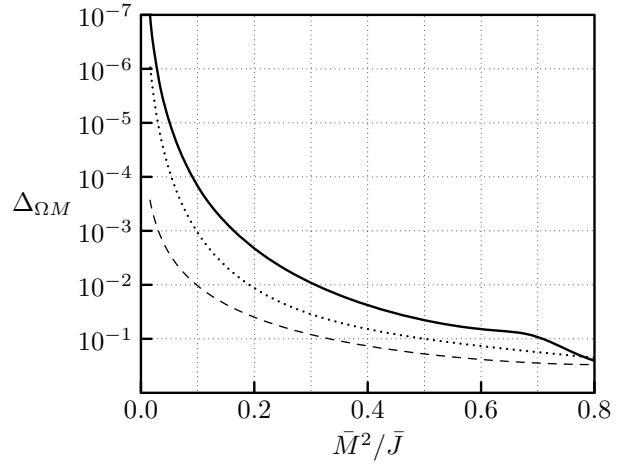


Figure 6. The relative errors for the sequence of rings corresponding to Fig. 4 ($\varrho_i/\varrho_o = 0.95$) determined by comparing them to the relativistic curve. The line types are the same as in Fig. 4.

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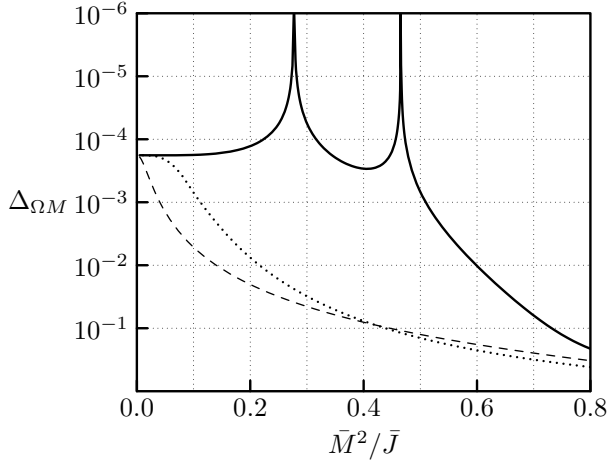


Figure 7. The relative errors for the sequence of rings corresponding to Fig. 5 ($\varrho_i/\varrho_o = 0.7$) determined by comparing them to the relativistic curve. The line types are the same as in Fig. 4. (The two singularities to be seen in the post-Newtonian curve with $d = d_M$ reflect nothing more than the fact that it crosses the correct relativistic curve twice.)

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APPENDIX A: THE COEFFICIENTS

In the Tables A1–A6 the post-Newtonian coefficients up to the order $q = 4$ are presented.

Table A1. Coefficients L_i up to the order $q = 4$.

i	L_i
1	$\frac{1}{48\sqrt{4\lambda+3}} (144\lambda^2 + 96\lambda d + 240\lambda + 24d + 119)$
2	0
3	$-\frac{1}{1152(4\lambda+3)^{3/2}} (3744\lambda d + 1728\lambda^2 d + 1439 + 7236\lambda + 6048\lambda^3 + 2076d + 11448\lambda^2)$
4	0

Table A2. Coefficients κ_{ik} up to the order $q = 4$.

i	k	κ_{ik}
1	1	0
2	1	0
2	2	$\frac{5}{2}\lambda^2 + \frac{15}{8}d\lambda + \frac{107}{48}\lambda + \frac{671}{1152} + \frac{15}{32}d$
3	1	0
3	2	0
3	3	$-\frac{9}{32}\lambda^2 + \frac{5}{32}d\lambda - \frac{1937}{3072}\lambda - \frac{19825}{73728} - \frac{65}{768}d$
4	1	0
4	2	$\frac{25}{8}\lambda^3 + \frac{39}{16}\lambda^2 + \frac{25}{8}d\lambda^2 - \frac{515}{1152}\lambda + \frac{315}{128}d\lambda - \frac{2093}{6912} - \frac{1115}{9216}d$
4	3	0
4	4	$\frac{75}{32}\lambda^3 + \frac{8111}{2304}\lambda^2 + \frac{375}{256}d\lambda^2 + \frac{2725}{2304}d\lambda + \frac{328073}{184320}\lambda + \frac{4190617}{13271040} + \frac{5885}{55296}d$

Table A3. Coefficients $N_{ik}^i(y)$ up to the order $q = 4$.

i	k	N_{ik}^i
0	0	$-12\lambda^2 - 54\lambda - 4d\lambda - \frac{245}{4} - 8d + (6\lambda + \frac{29}{2})y^2 - \frac{5}{4}y^4$
1	0	0
1	1	$(-6\lambda^2 - 16\lambda - 2d\lambda - \frac{47}{4} - d)y + (\frac{1}{2}\lambda + \frac{7}{2})y^3 - \frac{5}{12}y^5$
2	0	$\frac{5}{4}\lambda^2 + \frac{1}{2}d\lambda + \frac{133}{24}\lambda + \frac{3}{4}d + \frac{1123}{192} + (-\frac{3}{2}\lambda^2 - 6\lambda - \frac{1}{2}d\lambda - \frac{1}{4}d - \frac{461}{96})y^2 + (\frac{19}{16}\lambda + \frac{127}{64})y^4 - \frac{5}{32}y^6$
2	1	0
2	2	$(-\frac{17}{2}\lambda^2 - 2d\lambda - \frac{59}{3}\lambda - \frac{11159}{1152} - \frac{7}{24}d)y^2 + (\frac{67}{24}\lambda + \frac{217}{72})y^4 - \frac{65}{384}y^6$
3	0	0
3	1	$(\frac{85}{16}\lambda^2 + \frac{13}{8}d\lambda + \frac{343}{32}\lambda + \frac{11987}{2304} + \frac{91}{96}d)y + (-\frac{13}{4}\lambda^2 - \frac{7}{8}d\lambda - \frac{1727}{192}\lambda - \frac{25}{96}d - \frac{24167}{4608})y^3 + (\frac{257}{192}\lambda + \frac{1307}{576})y^5 - \frac{25}{128}y^7$
3	2	0
3	3	$(-\frac{97}{96}\lambda^2 - \frac{15}{32}d\lambda - \frac{14185}{4608}\lambda - \frac{50857}{36864} + \frac{5}{768}d)y^3 + (\frac{515}{768}\lambda + \frac{16465}{18432})y^5 - \frac{7}{96}y^7$
4	0	$-\frac{125}{16}\lambda^4 - \frac{75}{32}d\lambda^3 - \frac{1745}{64}\lambda^3 - \frac{47291}{1536}\lambda^2 - \frac{55}{16}d\lambda^2 - \frac{1097}{1536}d\lambda - \frac{623875}{55296}\lambda + \frac{1633}{9216}d - \frac{118327}{221184} + (\frac{75}{64}\lambda^3 + \frac{405}{256}\lambda^2 + \frac{3767}{3072}\lambda + \frac{13}{32}d\lambda + \frac{91}{384}d + \frac{9767}{12288})y^2 + (-\frac{39}{32}\lambda^2 - \frac{873}{256}\lambda - \frac{21}{64}d\lambda - \frac{25}{256}d - \frac{26059}{12288})y^4 + (\frac{135}{256}\lambda + \frac{1415}{1536})y^6 - \frac{675}{8192}y^8$
4	1	0
4	2	$(-\frac{55}{8}\lambda^3 - \frac{5}{2}d\lambda^2 - \frac{3517}{256}\lambda^2 - \frac{151}{128}d\lambda - \frac{246605}{36864}\lambda + \frac{534997}{884736} + \frac{5329}{9216}d)y^2 + (-\frac{217}{384}\lambda^2 - \frac{185}{384}d\lambda - \frac{6933}{2048}\lambda - \frac{985}{9216}d - \frac{1142287}{442368})y^4 + (\frac{2389}{3072}\lambda + \frac{95237}{73728})y^6 - \frac{175}{1536}y^8$
4	3	0
4	4	$(-\frac{2821}{2304}\lambda^2 - \frac{455}{2304}d\lambda - \frac{418169}{184320}\lambda + \frac{1001}{55296}d - \frac{10764521}{13271040})y^4 + (\frac{2611}{9216}\lambda + \frac{85229}{221184})y^6 - \frac{133}{4096}y^8$

Table A4. Coefficients F_{li} up to the order $q = 4$.

l	i	F_{li}
1	0	$-12\lambda - 24 - 4d$
1	1	0
1	2	$\frac{15}{16}$
1	3	0
1	4	$-\frac{125}{16}\lambda^3 - \frac{75}{32}d\lambda^2 - \frac{555}{32}\lambda^2 - \frac{17357}{1536}\lambda - \frac{125}{64}d\lambda - \frac{115483}{55296} - \frac{175}{512}d$
2	1	$\frac{5}{2}\lambda + \frac{10}{3} + d$
2	2	0
2	3	$\frac{135}{8}\lambda^2 + \frac{1939}{64}\lambda + \frac{75}{8}d\lambda + \frac{125}{32}d + \frac{18527}{1536}$
2	4	0
3	2	$-\frac{25}{4}\lambda^2 - \frac{15}{4}d\lambda - \frac{23}{2}\lambda - \frac{3}{2}d - \frac{2693}{576}$
3	3	0
3	4	$-\frac{55}{8}\lambda^3 - \frac{4375}{256}\lambda^2 - 5d\lambda^2 - \frac{168149}{12288}\lambda - \frac{1075}{192}d\lambda - \frac{3024601}{884736} - \frac{2645}{2304}d$
4	3	$\frac{143}{192}\lambda^2 + \frac{5}{16}d\lambda + \frac{11729}{9216}\lambda + \frac{4801}{8192} + \frac{23}{96}d$
4	4	0
5	4	$-\frac{175}{192}\lambda^3 - \frac{13427}{6912}\lambda^2 - \frac{125}{192}d\lambda^2 - \frac{2125}{3456}d\lambda - \frac{1505639}{1105920}\lambda - \frac{24750911}{79626240} - \frac{44165}{331776}d$

Table A5. Coefficients $w_{ik}^i(y)$ up to the order $q = 4$.

i	k	w_{ik}^i
0	0	$-4y^2 + 4 + 8\lambda$
1	0	0
1	1	$(12\lambda - 4)y - 3y^3$
2	0	$\frac{3}{2}\lambda + \frac{21}{8} + (9\lambda - 3)y^2 - \frac{15}{8}y^4$
2	1	0
2	2	$(10\lambda - \frac{35}{12})y^2 - \frac{7}{4}y^4$
3	0	0
3	1	$(-\frac{3}{8}\lambda + \frac{55}{16})y + (\frac{75}{4}\lambda - \frac{95}{16})y^3 - \frac{105}{32}y^5$
3	2	0
3	3	$(\frac{245}{48}\lambda - \frac{3395}{1152})y^3 - \frac{63}{64}y^5$
4	0	$\frac{25}{16}\lambda^3 + \frac{115}{96}\lambda^2 + \frac{1057}{2304}\lambda + \frac{115}{1152} + (-\frac{9}{32}\lambda + \frac{165}{64})y^2 + (\frac{375}{32}\lambda - \frac{475}{128})y^4 - \frac{245}{128}y^6$
4	1	0
4	2	$(\frac{5}{2}\lambda^2 + \frac{365}{128}\lambda + \frac{30815}{9216})y^2 + (\frac{945}{64}\lambda - \frac{2905}{512})y^4 - \frac{315}{128}y^6$
4	3	0
4	4	$(\frac{1645}{576}\lambda - \frac{27223}{13824})y^4 - \frac{693}{1280}y^6$

Table A6. Coefficients G_{li} up to the order $q = 4$.

l	i	G_{li}
1	0	-16
1	1	0
1	2	-12
1	3	0
1	4	$-\frac{25}{8}\lambda^2 - \frac{895}{48}\lambda - \frac{11305}{1152}$
2	1	6
2	2	0
2	3	$\frac{55}{2}\lambda + \frac{409}{24}$
2	4	0
3	2	$-5\lambda - \frac{19}{6}$
3	3	0
3	4	$-5\lambda^2 - \frac{995}{64}\lambda - \frac{34361}{4608}$
4	3	$\frac{55}{48}\lambda + \frac{869}{1152}$
4	4	0
5	4	$-\frac{25}{48}\lambda^2 - \frac{2195}{3456}\lambda - \frac{20819}{103680}$